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A New Eight Vertex Model and Higher Dimensional, Multiparameter Generalizations

Dedicated to the memory of Professor Jean Lascoux

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Abstract

We study statistical models, specifically transfer matrices corresponding to a multiparameter hierarchy of braid matrices of $(2n)^2 \times (2n)^2$ dimensions with $2n^2$ free parameters $(n = 1, 2, 3, \ldots)$. The simplest, 4×4 case is treated in detail. Powerful recursion relations are constructed giving the dependence on the spectral parameter θ of the eigenvalues of the transfer matrix explicitly at each level of coproduct sequence. A brief study of higher dimensional cases $(n \geq 2)$ is presented pointing out features of particular interest. Spin chain Hamiltonians are also briefly presented for the hierarchy. In a long final section basic results are recapitulated with systematic analysis of their contents. Our eight vertex 4×4 case is compared to standard six vertex and eight vertex models.

1 Introduction

In a previous paper [1] statistical models were presented starting from a class of multiparameter braid matrices of odd dimensions (Ref. 1 cites previous sources). This class of $N^2 \times N^2$ braid matrices for $N = 2n - 1$ $(n = 2, 3, \ldots)$ depends on $\frac{1}{2}(N+3)(N-1) = 2(n^2-1)$ parameters and has a basis of a *nested sequence* of projectors introduced before [2]. Recently this class has been extended to include *even* dimensions [3]. The even dimensional $(2n)^2 \times (2n)^2$ braid matrices depend, apart from the spectral parameter θ, on $2n^2$ free parameters. One such parameter can, as usual, be absorbed by a suitable choice of an overall normalization factor. Such a quadratic increase of free parameters with n (even or odd) is the most salient, unique feature of our models with their *nested sequence* basis.

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We start with a detailed study of the 4×4 case. This is our *new* eight vertex model. It should be compared to *exotic* eight vertex model based on the SØ3 bialgebra [4]. There negative Boltzmann weights were involved and were commented upon. Here, for suitable choices of parameters, one has all weights positive or zero for $\theta > 0$ or for an alternative choice for $\theta < 0$. This 4×4 case has another major distinguishing feature. It finds its place as the first step in an explicitly constructed multiparameter hierarchy. This is to be contrasted with the multistate generalization of six vertex model (Ref. 5 and the original sources cited there) in which the parametrization remains restricted to the six vertex level as the dimensions increases. We will present some remarkable general features conserved in the entire hierarchy. But here our study will be brief - a beginning. We hope to explore further elsewhere. We will also briefly present spin chain Hamiltonians for our class of solutions. Some aspects are better discussed after the constructions are presented. Such comments are reserved for the end.

2 Even dimensional multiparameter braid matrices

We briefly recapitulate the construction of Ref. 3, closely related to that for odd dimensions [1, 2]. The notation of Ref. 3 are maintained. Define the *nested sequence* of projectors

$$P_{ij}^{(\epsilon)} = \frac{1}{2}\left\{(ii) \otimes (jj) + (\bar{i}\bar{i}) \otimes (\bar{j}\bar{j}) + \epsilon\left[(i\bar{i}) \otimes (j\bar{j}) + (\bar{i}i) \otimes (\bar{j}j)\right]\right\}, \tag{2.1}$$

where $i, j \in \{1, \cdots, n\}$, $\epsilon = \pm$, $\bar{i} = 2n+1-i$, $\bar{j} = 2n+1-j$ and (ij) denotes the matrix with an unique non-zero element 1 on row i and column j. They provide a complete basis satisfying

$$P_{ij}^{(\epsilon)} P_{kl}^{(\epsilon')} = \delta_{ik}\delta_{jl}\delta_{\epsilon\epsilon'} P_{ij}^{(\epsilon)}, \qquad \text{and} \qquad \sum_{\epsilon=\pm}\sum_{i,j=1}^{n}\left(P_{ij}^{(\epsilon)} + P_{i\bar{j}}^{(\epsilon)}\right) = I_{(2n)^2\times(2n)^2}, \tag{2.2}$$

where $P_{i\bar{j}}^{(\epsilon)}$ is obtained by changing j with $\bar{j}$ in $P_{ij}^{(\epsilon)}$. The $(2n)^2 \times (2n)^2$ braid matrices

$$\hat{R}(\theta) = \sum_{\epsilon=\pm}\sum_{i,j=1}^{n} e^{m_{ij}^{(\epsilon)}\theta}\left(P_{ij}^{(\epsilon)} + P_{i\bar{j}}^{(\epsilon)}\right). \tag{2.3}$$

satisfy

$$\widehat{R}_{12}(\theta)\,\widehat{R}_{23}(\theta+\theta')\,\widehat{R}_{12}(\theta') = \widehat{R}_{23}(\theta')\,\widehat{R}_{12}(\theta+\theta')\,\widehat{R}_{23}(\theta), \tag{2.4}$$

where $\widehat{R}_{12} = \widehat{R} \otimes I$ and $\widehat{R}_{23} = I \otimes \widehat{R}$. Here, as for odd dimensions [1, 2] the free parameters enter as exponents $\left(e^{m_{ij}^{(\epsilon)}\theta}\right)$ and the basic constraint [1, 2] $m_{ij}^{(\epsilon)} = m_{i\bar{j}}^{(\epsilon)}$ has already been incorporated in (2.3). This leaves $2n^2$ free parameters. The simplest examples are:

• <u>$N = 2n = 2$</u>:

$$\hat{R}(\theta) = \begin{vmatrix} a_+ & 0 & 0 & a_- \\ 0 & a_+ & a_- & 0 \\ 0 & a_- & a_+ & 0 \\ a_- & 0 & 0 & a_+ \end{vmatrix}, \tag{2.5}$$

where

$$a_{\pm} = \frac{1}{2}\left(e^{m_{11}^{(+)}\theta} \pm e^{m_{11}^{(-)}\theta}\right). \tag{2.6}$$

- $\underline{N = 2n = 4}$:

$$\hat{R}(\theta) = \begin{vmatrix} D_{11} & 0 & 0 & A_{1\bar{1}} \\ 0 & D_{22} & A_{2\bar{2}} & 0 \\ 0 & A_{\bar{2}2} & D_{\bar{2}\bar{2}} & 0 \\ A_{\bar{1}1} & 0 & 0 & D_{\bar{1}\bar{1}} \end{vmatrix}, \tag{2.7}$$

with

$$D_{11} = D_{\bar{1}\bar{1}} = \begin{pmatrix} a_+ & 0 & 0 & 0 \\ 0 & b_+ & 0 & 0 \\ 0 & 0 & b_+ & 0 \\ 0 & 0 & 0 & a_+ \end{pmatrix}, \qquad D_{22} = D_{\bar{2}\bar{2}} = \begin{pmatrix} c_+ & 0 & 0 & 0 \\ 0 & d_+ & 0 & 0 \\ 0 & 0 & d_+ & 0 \\ 0 & 0 & 0 & c_+ \end{pmatrix},$$

$$A_{1\bar{1}} = A_{\bar{1}1} = \begin{pmatrix} 0 & 0 & 0 & a_- \\ 0 & 0 & b_- & 0 \\ 0 & b_- & 0 & 0 \\ a_- & 0 & 0 & 0 \end{pmatrix}, \qquad A_{2\bar{2}} = A_{\bar{2}2} = \begin{pmatrix} 0 & 0 & 0 & c_- \\ 0 & 0 & d_- & 0 \\ 0 & d_- & 0 & 0 \\ c_- & 0 & 0 & 0 \end{pmatrix} \tag{2.8}$$

and

$$\begin{aligned} a_{\pm} &= \frac{1}{2}\left(e^{m_{11}^{(+)}\theta} \pm e^{m_{11}^{(-)}\theta}\right), \qquad & b_{\pm} &= \frac{1}{2}\left(e^{m_{12}^{(+)}\theta} \pm e^{m_{12}^{(-)}\theta}\right), \\ c_{\pm} &= \frac{1}{2}\left(e^{m_{21}^{(+)}\theta} \pm e^{m_{21}^{(-)}\theta}\right), \qquad & d_{\pm} &= \frac{1}{2}\left(e^{m_{22}^{(+)}\theta} \pm e^{m_{22}^{(-)}\theta}\right) \end{aligned} \tag{2.9}$$

We will often write

$$D_{11} = (a_+, b_+, b_+, a_+)_{\text{diag.}}, \qquad A_{1\bar{1}} = (a_-, b_-, b_-, a_-)_{\text{anti-diag.}} \tag{2.10}$$

and so on. Generalization for $n > 2$ is entirely straightforward.

3 $\hat{R}$TT relations and transfer matrix ($n = 1$)

To simplify computations, taking out an overall factor a_+ we write $\hat{R}(\theta)$ of (2.5) as

$$\hat{R}(\mathbf{x}) = \begin{vmatrix} 1 & 0 & 0 & \mathbf{x} \\ 0 & 1 & \mathbf{x} & 0 \\ 0 & \mathbf{x} & 1 & 0 \\ \mathbf{x} & 0 & 0 & 1 \end{vmatrix}, \tag{3.1}$$

where

$$\mathbf{x} = \frac{e^{m_{11}^{(+)}\theta} - e^{m_{11}^{(-)}\theta}}{e^{m_{11}^{(+)}\theta} + e^{m_{11}^{(-)}\theta}} = \tanh(\mu\theta), \tag{3.2}$$

with $2\mu = m_{11}^{(+)} - m_{11}^{(-)}$. Note that

$$\left\{ m_{11}^{(+)} > m_{11}^{(-)},\, \theta > 0 \right\} \Longrightarrow \mathbf{x} > 0 \qquad \text{and} \qquad \left\{ m_{11}^{(+)} < m_{11}^{(-)},\, \theta < 0 \right\} \Longrightarrow \mathbf{x} > 0. \quad (3.3)$$

We consider these two domains separately, assuring nonnegative, real Boltzmann weights. For purely imaginary parameters $\mathbf{i}m_{11}^{(\pm)}$ ($m_{11}^{(\pm)}$ real)

$$\mathbf{x} = \mathbf{i} \tan (\mu\theta) \,, \tag{3.4}$$

with a normalization factor $\cos \mu\theta$, one obtains a unitary braid matrix

$$\hat{R} (\mathbf{x})^{+} \hat{R} (\mathbf{x}) = I. \tag{3.5}$$

This has been pointed out in sec. 3 of Ref. 3 citing Ref. 6. We continue to study here real matrices.

The basic $\hat{R}\mathbf{TT}$ equation can now be written as

$$\hat{R} (\mathbf{x}'') (\mathbf{T} (\mathbf{x}) \otimes \mathbf{T} (\mathbf{x}')) = (\mathbf{T} (\mathbf{x}') \otimes \mathbf{T} (\mathbf{x})) \hat{R} (\mathbf{x}'') \,, \tag{3.6}$$

where

$$\mathbf{x} = \tanh (\mu\theta) \,, \qquad \mathbf{x}' = \tanh (\mu\theta') \,, \qquad \mathbf{x}'' = \frac{\mathbf{x} - \mathbf{x}'}{1 - \mathbf{x}\mathbf{x}'} = \tanh \mu (\theta - \theta') \,. \tag{3.7}$$

We denote $\mathbf{T} = \mathbf{T} (\mathbf{x})$, $\mathbf{T}' = \mathbf{T} (\mathbf{x}')$, $\hat{R}'' = \hat{R} (\mathbf{x}'')$ and the 4 blocks of $\mathbf{T}$ as

$$\mathbf{T} = \begin{vmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{vmatrix} \equiv \begin{vmatrix} A & B \\ C & D \end{vmatrix}. \tag{3.8}$$

Defining the 2×2 matrices

$$I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \qquad K = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \tag{3.9}$$

(3.6) is now (with $A = A (\mathbf{x})$, $A' = A (\mathbf{x}')$, etc.)

$$(I \otimes I + \mathbf{x}'' K \otimes K) \begin{vmatrix} A & B \\ C & D \end{vmatrix} \otimes \begin{vmatrix} A' & B' \\ C' & D' \end{vmatrix} = \begin{vmatrix} A' & B' \\ C' & D' \end{vmatrix} \otimes \begin{vmatrix} A & B \\ C & D \end{vmatrix} (I \otimes I + \mathbf{x}'' K \otimes K) \,. \tag{3.10}$$

A detailed study of (3.10) is given in Appendix. Recursion relations will be extracted from it and implemented to construct the spectrum of the eigenvalues of the transfer matrix in sec. 5. The transfer matrix $(A + D)_r \equiv A_r + D_r$ is obtained by starting with fundamental 2×2 blocks (for $n = 1$) and constructing $2^r \times 2^r$ blocks using standard prescriptions (coproduct rules - see sec. 5). The starting point, the 2×2 blocks for $r = 1$, is provided by the Yang-Baxter matrix corresponding to (3.1), namely

$$R (\mathbf{x}) = \mathrm{P}\hat{R} (\mathbf{x}) = \begin{vmatrix} 1 & 0 & 0 & \mathbf{x} \\ 0 & \mathbf{x} & 1 & 0 \\ 0 & 1 & \mathbf{x} & 0 \\ \mathbf{x} & 0 & 0 & 1 \end{vmatrix} \equiv \begin{vmatrix} A_1 & B_1 \\ C_1 & D_1 \end{vmatrix}, \tag{3.11}$$

where P is the standard 4×4 permutation matrix. Then one uses the coproduct rule

$$\left(\mathbf{T}_{ij}\right)_{r+1} = \sum_{k} \left(\mathbf{T}_{ik}\right)_1 \otimes \left(\mathbf{T}_{kj}\right)_r . \tag{3.12}$$

Such a construction guarantees the commutativity of the transfer matrix , i.e.

$$\left[\mathbf{T}\left(\mathbf{x}\right), \mathbf{T}\left(\mathbf{x}'\right)\right] = 0 \tag{3.13}$$

quite generally and reducing for our present case to

$$\left[\left(A + D\right)_r, \left(A + D\right)'_r\right] = 0 \tag{3.14}$$

for all r. This is the basic ingredient of exactly solvable statistical models [7]. The trace and the highest eigenvalue of $\mathbf{T}\left(\mathbf{x}\right)$ provide significant features of the corresponding models. We will obtain these quite simply and generally for our class. For $n > 1$ certain essential features will be briefly presented in sec. 6.

4 Eigenfunctions and eigenvalues of the transfer matrix

The next essential step is the construction of eigenstates and eigenvalues of the transfer matrix. In this section we display the results for $r = 1, 2, 3, 4$. They provide explicit examples of the iterative structure of the transfer matrices $\mathbf{T}_r$ to be derived in the next section. Moreover they illustrate how multiplets involving r-th roots of unity and possible multiplicities of them combine to provide a complete basis of mutually orthogonal eigenstates of $\mathbf{T}_r$ spanning the base space. Referring back to them one grasps better the full content of the general formalism of sec. 5. We start with the notations of sec. 3.

• $\underline{r = 1}$:

$$\mathbf{T}_1 = \left(A + D\right)_1 = \left(1 + \mathbf{x}\right) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \qquad \left(A - D\right)_1 = \left(1 - \mathbf{x}\right) \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}. \tag{4.1}$$

The eigenstates are evidently

$$|1\rangle \equiv \begin{vmatrix} 1 \\ 0 \end{matrix}\right\rangle, \qquad |\overline{1}\rangle \equiv \left|\begin{matrix} 0 \\ 1 \end{matrix}\right\rangle \tag{4.2}$$

with

$$\mathbf{T}_1 \left(|1\rangle, |\overline{1}\rangle\right) = \left(1 + \mathbf{x}\right) \left(|1\rangle, |\overline{1}\rangle\right) \tag{4.3}$$

and

$$\mathrm{Tr}\left(\mathbf{T}_1\right) = 2\left(1 + \mathbf{x}\right). \tag{4.4}$$

• $\underline{r = 2}$:

$$\mathbf{T}_2 = \left(A + D\right)_2 = \left(1 + \mathbf{x}\right)^2 \mathbf{X}_{(2,0)} + \left(1 - \mathbf{x}\right)^2 \mathbf{X}_{(0,2)}, \tag{4.5}$$

where

$$\mathbf{X}_{(2,0)} = \frac{1}{2}\begin{vmatrix} I_{(1)} & K_{(1)} \\ K_{(1)} & I_{(1)} \end{vmatrix} \otimes \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \qquad \mathbf{X}_{(0,2)} = \frac{1}{2}\begin{vmatrix} I_{(1)} & K_{(1)} \\ -K_{(1)} & -I_{(1)} \end{vmatrix} \otimes \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \tag{4.6}$$

with $I_{(1)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ and $K_{(1)} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$. One has

$$\mathbf{X}_{(2,0)}\mathbf{X}_{(0,2)} = \mathbf{X}_{(0,2)}\mathbf{X}_{(2,0)} = 0. \tag{4.7}$$

We have anticipated the iterative structure of sec. 5. That $\mathbf{T}_2$ obtained by a straightforward use of coproduct rules as

$$\mathbf{T}_2 = \begin{vmatrix} 1+\mathbf{x}^2 & 0 & 0 & 2\mathbf{x} \\ 0 & 2\mathbf{x} & (1+\mathbf{x})^2 & 0 \\ 0 & (1+\mathbf{x})^2 & 2\mathbf{x} & 0 \\ 2\mathbf{x} & 0 & 0 & 1+\mathbf{x}^2 \end{vmatrix} \tag{4.8}$$

can be expressed as (4.5) on a basis satisfying (4.7) is the central lesson. Denote

$$\begin{vmatrix} 1 \\ 0 \end{matrix}\right\rangle \otimes \left|\begin{matrix} 1 \\ 0 \end{matrix}\right\rangle \equiv |11\rangle\,, \qquad \left|\begin{matrix} 0 \\ 1 \end{matrix}\right\rangle \otimes \left|\begin{matrix} 0 \\ 1 \end{matrix}\right\rangle \equiv |\bar{1}\bar{1}\rangle\,, \qquad \left|\begin{matrix} 1 \\ 0 \end{matrix}\right\rangle \otimes \left|\begin{matrix} 0 \\ 1 \end{matrix}\right\rangle \equiv |1\bar{1}\rangle\,, \qquad \left|\begin{matrix} 0 \\ 1 \end{matrix}\right\rangle \otimes \left|\begin{matrix} 1 \\ 0 \end{matrix}\right\rangle \equiv |\bar{1}1\rangle\,. \tag{4.9}$$

For $r > 1$ the order of the indices $(1, \bar{1})$ indicates the structure of the tensor product. For example, $|1\bar{1}1\rangle = \left|\begin{matrix} 1 \\ 0 \end{matrix}\right\rangle \otimes \left|\begin{matrix} 0 \\ 1 \end{matrix}\right\rangle \otimes \left|\begin{matrix} 1 \\ 0 \end{matrix}\right\rangle$. One obtains, with upper or lower signs,

$$\mathbf{T}_2\left(|11\rangle \pm |\bar{1}\bar{1}\rangle\right) = (1 \pm \mathbf{x})^2\left(|11\rangle \pm |\bar{1}\bar{1}\rangle\right), \qquad \mathbf{T}_2\left(|1\bar{1}\rangle \pm |\bar{1}1\rangle\right) = \pm(1 \pm \mathbf{x})^2\left(|1\bar{1}\rangle \pm |\bar{1}1\rangle\right) \tag{4.10}$$

and

$$\mathrm{Tr}\left(\mathbf{T}_2\right) = 2\left(1+\mathbf{x}\right)^2. \tag{4.11}$$

The eigenfunctions of $\mathbf{X}_{(2,0)}$ (resp. $\mathbf{X}_{(0,2)}$) are annihilated by $\mathbf{X}_{(0,2)}$ (resp. $\mathbf{X}_{(2,0)}$) consistently with (4.7). Finally, again anticipating sec. 5,

$$(A-D)_2 = \frac{1}{2}(1+\mathbf{x})\begin{vmatrix} I_{(1)} & -K_{(1)} \\ -K_{(1)} & I_{(1)} \end{vmatrix} \otimes (A-D)_1 + \frac{1}{2}(1-\mathbf{x})\begin{vmatrix} I_{(1)} & -K_{(1)} \\ K_{(1)} & -I_{(1)} \end{vmatrix} \otimes (A+D)_1\,. \tag{4.12}$$

• <u>$r=3$</u>: From this stage onwards one can better appreciate the role of multiplets involving roots of unity in constructing eigenstates. One obtains by implementing coproducts in step $(r=2) \longrightarrow (r=3)$,

$$\mathbf{T}_3 = (A+D)_3 = (1+\mathbf{x})^3\,\mathbf{X}_{(3,0)} + (1+\mathbf{x})(1-\mathbf{x})^2\mathbf{X}_{(1,2)}, \tag{4.13}$$

where the 8×8 matrices satisfy

$$\mathbf{X}_{(3,0)}\mathbf{X}_{(1,2)} = \mathbf{X}_{(1,2)}\mathbf{X}_{(3,0)} = 0 \tag{4.14}$$

and they are $\mathbf{x}$-independent. They are easy to obtain but will not, for brevity, be presented explicitly. The base space splits up into 4-dim. ones, closed under the action of $\mathbf{T}_3$ and are characterized by even (odd) number of the index 1 (considering zero as even) respectively. Define

$$
\begin{aligned}
|e_1\rangle &= |\bar{1}\bar{1}\bar{1}\rangle + |\bar{1}11\rangle + |1\bar{1}1\rangle + |11\bar{1}\rangle \,, \\
|e_2\rangle &= -3\,|\bar{1}\bar{1}\bar{1}\rangle + |\bar{1}11\rangle + |1\bar{1}1\rangle + |11\bar{1}\rangle \,, \\
|e_3\rangle &= |\bar{1}11\rangle + \omega\,|1\bar{1}1\rangle + \omega^2\,|11\bar{1}\rangle \,, \\
|e_4\rangle &= |\bar{1}11\rangle + \omega^2\,|1\bar{1}1\rangle + \omega\,|11\bar{1}\rangle \,,
\end{aligned}
\tag{4.15}
$$

where $\omega = e^{\mathbf{i}\frac{2\pi}{3}}$ ($\omega^3 = 1\,\text{and}\,1 + \omega + \omega^2 = 0$). One has $\langle e_i|e_j\rangle = 0,\ i \neq j$, where $\langle e_i|$ denotes the transform of $|e_i\rangle$ with conjugated coefficients. Denote also

$$
\begin{aligned}
|o_1\rangle &= |111\rangle + |1\bar{1}\bar{1}\rangle + |\bar{1}1\bar{1}\rangle + |\bar{1}\bar{1}1\rangle \,, \\
|o_2\rangle &= -3\,|111\rangle + |1\bar{1}\bar{1}\rangle + |\bar{1}1\bar{1}\rangle + |\bar{1}\bar{1}1\rangle \,, \\
|o_3\rangle &= |1\bar{1}\bar{1}\rangle + \omega\,|\bar{1}1\bar{1}\rangle + \omega^2\,|\bar{1}\bar{1}1\rangle \,, \\
|o_4\rangle &= |1\bar{1}\bar{1}\rangle + \omega^2\,|\bar{1}1\bar{1}\rangle + \omega\,|\bar{1}\bar{1}1\rangle \,.
\end{aligned}
\tag{4.16}
$$

The states $\{|e_i\rangle\,, |o_j\rangle\}$ form a complete basis of orthogonal states. Consistently with (4.13), (4.14) one obtains

$$
\begin{aligned}
&\mathbf{T}_3\,|e_1\rangle = (1+\mathbf{x})^3\,|e_1\rangle \,, \\
&\mathbf{T}_3\,(|e_2\rangle\,, |e_3\rangle\,, |e_4\rangle) = (1+\mathbf{x})\,(1-\mathbf{x})^2\,\big(|e_2\rangle\,, \omega\,|e_3\rangle\,, \omega^2\,|e_4\rangle\big)\,, \\
&\mathbf{T}_3\,|o_1\rangle = (1+\mathbf{x})^3\,|o_1\rangle \,, \\
&\mathbf{T}_3\,(|o_2\rangle\,, |o_3\rangle\,, |o_4\rangle) = (1+\mathbf{x})\,(1-\mathbf{x})^2\,\big(|o_2\rangle\,, \omega\,|o_3\rangle\,, \omega^2\,|o_4\rangle\big)
\end{aligned}
\tag{4.17}
$$

and, finally,

$$
\mathrm{Tr}\,(\mathbf{T}_3) = 2\,(1+\mathbf{x})^3 + 2\,(1+\mathbf{x})\,(1-\mathbf{x})^2\,\big(1+\omega+\omega^2\big) = 2\,(1+\mathbf{x})^3\,. \tag{4.18}
$$

• $\underline{r=4}$: Now

$$
\mathbf{T}_4 = (A+D)_4 = (1+\mathbf{x})^4\,\mathbf{X}_{(4,0)} + (1+\mathbf{x})^2\,(1-\mathbf{x})^2\,\mathbf{X}_{(2,2)} + (1-\mathbf{x})^4\,\mathbf{X}_{(0,4)}, \tag{4.19}
$$

where the 16×16 constant matrices satisfy

$$
\mathbf{X}_{(4,0)}\mathbf{X}_{(2,2)} = \mathbf{X}_{(4,0)}\mathbf{X}_{(0,4)} = \mathbf{X}_{(2,2)}\mathbf{X}_{(0,4)} = \mathbf{X}_{(2,2)}\mathbf{X}_{(4,0)} = \mathbf{X}_{(0,4)}\mathbf{X}_{(4,0)} = \mathbf{X}_{(0,4)}\mathbf{X}_{(2,2)} = 0 \tag{4.20}
$$

as consequence of recursion relations involved in $(A \pm D)_3 \longrightarrow (A \pm D)_4$. The $\mathbf{X}$'s are obtained fairly easily. Now the even and odd subspaces are 8-dim. We display the eigenstates explicitly to illustrate a new feature. Now $r = 4$ is not a prime number and 4-th roots and square roots of unity (corresponding to $r = 2 \times 2$) both contribute multiplets

involving respective coefficients $(1, \mathbf{i}, -1, -\mathbf{i})$ and $(1, -1)$ (For $r = 6$ one would have thus 2-plets, 3-plets and 6-plets). The $|e\rangle$ and $|o\rangle$ spaces have the orthogonal bases

$$
\begin{aligned}
|e_1\rangle &= |1111\rangle + |11\bar{1}\bar{1}\rangle + |1\bar{1}1\bar{1}\rangle + |1\bar{1}\bar{1}1\rangle + |\bar{1}11\bar{1}\rangle + |\bar{1}1\bar{1}1\rangle + |\bar{1}\bar{1}11\rangle + |\bar{1}\bar{1}\bar{1}\bar{1}\rangle\,,\\
|e_2\rangle &= |1111\rangle - |11\bar{1}\bar{1}\rangle + |1\bar{1}1\bar{1}\rangle - |1\bar{1}\bar{1}1\rangle - |\bar{1}11\bar{1}\rangle + |\bar{1}1\bar{1}1\rangle - |\bar{1}\bar{1}11\rangle + |\bar{1}\bar{1}\bar{1}\bar{1}\rangle\,,\\
|e_3\rangle &= |1111\rangle - |\bar{1}\bar{1}\bar{1}\bar{1}\rangle\,,\\
|e_4\rangle &= |1111\rangle - |1\bar{1}1\bar{1}\rangle - |\bar{1}1\bar{1}1\rangle + |\bar{1}\bar{1}\bar{1}\bar{1}\rangle\,,\\
|e_5\rangle &= |11\bar{1}\bar{1}\rangle - |1\bar{1}\bar{1}1\rangle - |\bar{1}11\bar{1}\rangle + |\bar{1}\bar{1}11\rangle\,,\\
|e_6\rangle &= |1\bar{1}1\bar{1}\rangle - |\bar{1}1\bar{1}1\rangle\,,\\
|e_7\rangle &= |11\bar{1}\bar{1}\rangle - \mathbf{i}\,|1\bar{1}\bar{1}1\rangle + \mathbf{i}\,|\bar{1}11\bar{1}\rangle - |\bar{1}\bar{1}11\rangle\,,\\
|e_8\rangle &= |11\bar{1}\bar{1}\rangle + \mathbf{i}\,|1\bar{1}\bar{1}1\rangle - \mathbf{i}\,|\bar{1}11\bar{1}\rangle - |\bar{1}\bar{1}11\rangle\,,
\end{aligned}
\tag{4.21}
$$

and

$$
\begin{aligned}
|o_1\rangle &= |111\bar{1}\rangle + |11\bar{1}1\rangle + |1\bar{1}11\rangle + |\bar{1}111\rangle + |1\bar{1}\bar{1}\bar{1}\rangle + |\bar{1}1\bar{1}\bar{1}\rangle + |\bar{1}\bar{1}1\bar{1}\rangle + |\bar{1}\bar{1}\bar{1}1\rangle\,,\\
|o_2\rangle &= |111\bar{1}\rangle - |11\bar{1}1\rangle + |1\bar{1}11\rangle - |\bar{1}111\rangle + |\bar{1}\bar{1}\bar{1}1\rangle - |\bar{1}\bar{1}1\bar{1}\rangle + |\bar{1}1\bar{1}\bar{1}\rangle - |1\bar{1}\bar{1}\bar{1}\rangle\,,\\
|o_3\rangle &= |111\bar{1}\rangle + |11\bar{1}1\rangle + |1\bar{1}11\rangle + |\bar{1}111\rangle - |\bar{1}\bar{1}\bar{1}1\rangle - |\bar{1}\bar{1}1\bar{1}\rangle - |\bar{1}1\bar{1}\bar{1}\rangle - |1\bar{1}\bar{1}\bar{1}\rangle\,,\\
|o_4\rangle &= |111\bar{1}\rangle - |11\bar{1}1\rangle + |1\bar{1}11\rangle - |\bar{1}111\rangle - |\bar{1}\bar{1}\bar{1}1\rangle + |\bar{1}\bar{1}1\bar{1}\rangle - |\bar{1}1\bar{1}\bar{1}\rangle + |1\bar{1}\bar{1}\bar{1}\rangle\,,\\
|o_5\rangle &= |111\bar{1}\rangle - \mathbf{i}\,|11\bar{1}1\rangle - |1\bar{1}11\rangle + \mathbf{i}\,|\bar{1}111\rangle\,,\\
|o_6\rangle &= |1\bar{1}\bar{1}\bar{1}\rangle + \mathbf{i}\,|\bar{1}1\bar{1}\bar{1}\rangle - |\bar{1}\bar{1}1\bar{1}\rangle - \mathbf{i}\,|\bar{1}\bar{1}\bar{1}1\rangle\,,\\
|o_7\rangle &= |111\bar{1}\rangle + \mathbf{i}\,|11\bar{1}1\rangle - |1\bar{1}11\rangle - \mathbf{i}\,|\bar{1}111\rangle\,,\\
|o_8\rangle &= |1\bar{1}\bar{1}\bar{1}\rangle - \mathbf{i}\,|\bar{1}1\bar{1}\bar{1}\rangle - |\bar{1}\bar{1}1\bar{1}\rangle + \mathbf{i}\,|\bar{1}\bar{1}\bar{1}1\rangle\,.
\end{aligned}
\tag{4.22}
$$

Define

$$
\mathbf{T}_4\,|e_i\rangle = \upsilon_i^{(e)}\,|e_i\rangle\,, \qquad \mathbf{T}_4\,|o_i\rangle = \upsilon_i^{(o)}\,|o_i\rangle\,, \qquad i = 1, \ldots, 8. \tag{4.23}
$$

Then one obtains, in order,

$$
\begin{aligned}
\upsilon_1^{(e)}, \ldots, \upsilon_8^{(e)} &= (1+\mathbf{x})^4\,, (1-\mathbf{x})^4\,, (1-\mathbf{x})^2\,(1+\mathbf{x})^2\,(1, 1, -1, -1, \mathbf{i}, -\mathbf{i})\,,\\
\upsilon_1^{(o)}, \ldots, \upsilon_8^{(o)} &= (1+\mathbf{x})^4\,, -(1-\mathbf{x})^4\,, (1-\mathbf{x})^2\,(1+\mathbf{x})^2\,(1, -1, \mathbf{i}, \mathbf{i}, -\mathbf{i}, -\mathbf{i})
\end{aligned}
\tag{4.24}
$$

and

$$
\mathrm{Tr}\,(\mathbf{T}_4) = 2\,(1+\mathbf{x})^4\,. \tag{4.25}
$$

5 Relating (A, B, C, D) for all r and constructing iteratively the eigenvalue spectrum

We consider below exclusively the 4×4 case. Coproduct rules lead, for $N = 2$, to the recursion relations

$$
\begin{aligned}
A_{r+1} &= A_1 \otimes A_r + B_1 \otimes C_r, \qquad & D_{r+1} &= D_1 \otimes D_r + C_1 \otimes B_r,\\
B_{r+1} &= A_1 \otimes B_r + B_1 \otimes D_r, \qquad & C_{r+1} &= C_1 \otimes A_r + D_1 \otimes C_r,
\end{aligned}
\tag{5.1}
$$

where

$$A_1 = \begin{vmatrix} 1 & 0 \\ 0 & \mathbf{x} \end{vmatrix}, \qquad D_1 = \begin{vmatrix} \mathbf{x} & 0 \\ 0 & 1 \end{vmatrix}, \qquad B_1 = \begin{vmatrix} 0 & \mathbf{x} \\ 1 & 0 \end{vmatrix}, \qquad C_1 = \begin{vmatrix} 0 & 1 \\ \mathbf{x} & 0 \end{vmatrix}. \tag{5.2}$$

Thus

$$A_{r+1} = \begin{vmatrix} A_r & \mathbf{x}C_r \\ C_r & \mathbf{x}A_r \end{vmatrix}, \ D_{r+1} = \begin{vmatrix} \mathbf{x}D_r & B_r \\ \mathbf{x}B_r & D_r \end{vmatrix}, \ B_{r+1} = \begin{vmatrix} B_r & \mathbf{x}D_r \\ D_r & \mathbf{x}B_r \end{vmatrix}, \ C_{r+1} = \begin{vmatrix} \mathbf{x}C_r & A_r \\ \mathbf{x}A_r & C_r \end{vmatrix}. \tag{5.3}$$

Denote $I \equiv I_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, $K \equiv K_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ and for p factors $I_{(p)} = I \otimes I \otimes \ldots \otimes I$, $K^{(p)} = K \otimes K \otimes \ldots \otimes K$. Starting from $B_1 = KA_1$, $C_1 = KD_1$, $D_1 = KA_1K$ and iterating one obtains

$$\begin{aligned} B_r &= \left(I_{(r-1)} \otimes K\right) A_r, \qquad C_r = \left(I_{(r-1)} \otimes K\right) D_r, \\ D_r &= K^{(r)} A_r K^{(r)}, \qquad A_r = K^{(r)} D_r K^{(r)}. \end{aligned} \tag{5.4}$$

Using $K^{(r)} \left(I_{(r-1)} \otimes K\right) = K^{(r-1)} \otimes I$ one can express $(A,B,C,D)_r$ in term of any one of them. In particular,

$$(B_r \pm C_r) = \left(I_{(r-1)} \otimes K\right) (A_r \pm D_r)\,, \tag{5.5}$$

where

$$\left(I_{(r-1)} \otimes K\right) = (K, K, \ldots, K)_{\text{diag.}} \equiv K_{(r)} \tag{5.6}$$

Thus

$$(B_3 \pm C_3) = \begin{vmatrix} K & 0 & 0 & 0 \\ 0 & K & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & K \end{vmatrix} (A_3 \pm D_3)\,. \tag{5.7}$$

Thus exchanging members of successive pairs of rows $[(1,2),(3,4),\ldots,(2p-1,2p),\ldots]_{(\text{rows})}$ one obtains $(B_r \pm C_r)$ from $(A_r \pm D_r)$. The following recursion relations are implied (with the definition (5.6) of $K_{(r)}$)

$$\begin{aligned} (A+D)_{r+1} &= \begin{vmatrix} (A+\mathbf{x}D)_r & (B+\mathbf{x}C)_r \\ (\mathbf{x}B+C)_r & (\mathbf{x}A+D)_r \end{vmatrix} \\ &= \frac{1}{2}(1+\mathbf{x}) \begin{vmatrix} I_{(r)} & K_{(r)} \\ K_{(r)} & I_{(r)} \end{vmatrix} \begin{vmatrix} (A+D)_r & 0 \\ 0 & (A+D)_r \end{vmatrix} \\ &+ \frac{1}{2}(1-\mathbf{x}) \begin{vmatrix} I_{(r)} & K_{(r)} \\ -K_{(r)} & -I_{(r)} \end{vmatrix} \begin{vmatrix} (A-D)_r & 0 \\ 0 & (A-D)_r \end{vmatrix} \end{aligned} \tag{5.8}$$

$$\begin{aligned} (A-D)_{r+1} &= \frac{1}{2}(1+\mathbf{x}) \begin{vmatrix} I_{(r)} & -K_{(r)} \\ -K_{(r)} & I_{(r)} \end{vmatrix} \begin{vmatrix} (A-D)_r & 0 \\ 0 & (A-D)_r \end{vmatrix} \\ &+ \frac{1}{2}(1-\mathbf{x}) \begin{vmatrix} I_{(r)} & -K_{(r)} \\ K_{(r)} & -I_{(r)} \end{vmatrix} \begin{vmatrix} (A+D)_r & 0 \\ 0 & (A+D)_r \end{vmatrix} \end{aligned} \tag{5.9}$$

The signification of these relations concerning eigenvalues and why we display also $(A-D)_{r+1}$ will be explained below.

Let us introduce at this point the general possibilities

$$
\begin{aligned}
A_{r_1+r_2} &= A_{r_1} \otimes A_{r_2} + B_{r_1} \otimes C_{r_2}, \qquad & D_{r_1+r_2} &= D_{r_1} \otimes D_{r_2} + C_{r_1} \otimes B_{r_2}, \\
B_{r_1+r_2} &= A_{r_1} \otimes B_{r_2} + B_{r_1} \otimes D_{r_2}, \qquad & C_{r_1+r_2} &= C_{r_1} \otimes A_{r_2} + D_{r_1} \otimes C_{r_2}.
\end{aligned} \tag{5.10}
$$

Since the sequence for odd and even r have some distinct typical features a *two-step* iteration can be of interest for $(A \pm D)$. One has (in evident notations)

$$
\begin{aligned}
2\,(A+D)_{r+2} &= (A+D)_2 \otimes (A+D)_r + (A-D)_2 \otimes (A-D)_r \\
&\quad + (B+C)_2 \otimes (B+C)_r - (B-C)_2 \otimes (B-C)_r\,,
\end{aligned} \tag{5.11}
$$

$$
\begin{aligned}
2\,(A-D)_{r+2} &= (A+D)_2 \otimes (A-D)_r + (A-D)_2 \otimes (A+D)_r \\
&\quad - (B+C)_2 \otimes (B-C)_r + (B-C)_2 \otimes (B+C)_r
\end{aligned} \tag{5.12}
$$

leading to

$$
\begin{aligned}
&4\,(A+D)_{r+2} = \\
&(1+\mathbf{x})^2 \begin{vmatrix} I_{(r)} & K_{(r)} & K_{(r)} & I_{(r)} \\ K_{(r)} & I_{(r)} & I_{(r)} & K_{(r)} \\ K_{(r)} & I_{(r)} & I_{(r)} & K_{(r)} \\ I_{(r)} & K_{(r)} & K_{(r)} & I_{(r)} \end{vmatrix} \begin{vmatrix} (A+D)_r & 0 & 0 & 0 \\ 0 & (A+D)_r & 0 & 0 \\ 0 & 0 & (A+D)_r & 0 \\ 0 & 0 & 0 & (A+D)_r \end{vmatrix} \\
&+ (1-\mathbf{x})^2 \begin{vmatrix} I_{(r)} & -K_{(r)} & K_{(r)} & -I_{(r)} \\ K_{(r)} & -I_{(r)} & I_{(r)} & -K_{(r)} \\ -K_{(r)} & I_{(r)} & -I_{(r)} & K_{(r)} \\ -I_{(r)} & K_{(r)} & -K_{(r)} & I_{(r)} \end{vmatrix} \begin{vmatrix} (A+D)_r & 0 & 0 & 0 \\ 0 & (A+D)_r & 0 & 0 \\ 0 & 0 & (A+D)_r & 0 \\ 0 & 0 & 0 & (A+D)_r \end{vmatrix} \\
&+ (1+\mathbf{x})\,(1-\mathbf{x}) \begin{vmatrix} I_{(r)} & 0 & K_{(r)} & 0 \\ -K_{(r)} & 0 & -I_{(r)} & 0 \\ 0 & I_{(r)} & 0 & K_{(r)} \\ 0 & -K_{(r)} & 0 & -I_{(r)} \end{vmatrix} \begin{vmatrix} (A-D)_r & 0 & 0 & 0 \\ 0 & (A-D)_r & 0 & 0 \\ 0 & 0 & (A-D)_r & 0 \\ 0 & 0 & 0 & (A-D)_r \end{vmatrix}
\end{aligned} \tag{5.13}
$$

From (5.12) one obtains an analogous result for $(A-D)_{r+2}$.

We now extract one fundamental consequence of the recursion relations (5.8-13). Starting with $(A+D)_1 = (1+\mathbf{x}) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ and $(A-D)_1 = (1-\mathbf{x}) \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$ these relations imply (with $\mathbf{x}$-independent matrices $\mathbf{X}$)

$$
\begin{aligned}
(A+D)_r &= (1+\mathbf{x})^r\, \mathbf{X}_{(r,0)} + (1+\mathbf{x})^{r-2}\,(1-\mathbf{x})^2\, \mathbf{X}_{(r-2,2)} \\
&\quad + \cdots + (1+\mathbf{x})^2\,(1-\mathbf{x})^{r-2}\, \mathbf{X}_{(2,r-2)} + (1-\mathbf{x})^r\, \mathbf{X}_{(0,r)}
\end{aligned} \tag{5.14}
$$

for even r and

$$
(A+D)_r = (1+\mathbf{x})^r\, \mathbf{X}_{(r,0)} + \cdots + (1+\mathbf{x})\,(1-\mathbf{x})^{r-1}\, \mathbf{X}_{(1,r-1)} \tag{5.15}
$$

for odd r, powers of $(1-\mathbf{x})$ being always even in both cases. Correspondingly (again with $\mathbf{x}$-independent $2^r \times 2^r$ matrices $\mathbf{Y}$)

$$
\begin{aligned}
(A-D)_r &= (1+\mathbf{x})^{r-1}\,(1-\mathbf{x})\, \mathbf{Y}_{(r-1,1)} + (1+\mathbf{x})^{r-3}\,(1-\mathbf{x})^3\, \mathbf{Y}_{(r-3,3)} + \cdots + \\
&\quad (1+\mathbf{x})^{\delta}\,(1-\mathbf{x})^{r-\delta}\, \mathbf{Y}_{(\delta,r-\delta)},
\end{aligned} \tag{5.16}
$$

where $\delta = \frac{1}{2}\left(1 + (-1)^r\right)$. There being here only odd powers of $(1 - \mathbf{x})$. When these series are inserted in (5.8) and (5.9) the features above are conserved.

From the general constraints (see Appendix)

$$(A \pm D)\,(A' \pm D') = (A' \pm D')\,(A \pm D)\,, \tag{5.17}$$

where $A = A(\mathbf{x})$, $A' = A(\mathbf{x}')$, etc. it already follows that the $\mathbf{X}$ (resp. $\mathbf{Y}$) must commute. Thus with $\mathbf{X}_{(r-2p,2p)} \equiv \mathbf{X}_{(p)}$, $\mathbf{Y}_{(r-2p-1,2p+1)} \equiv \mathbf{Y}_{(p)}$ one must have, for all (p, q)

$$\left[\mathbf{X}_{(p)}, \mathbf{X}_{(q)}\right] = \left[\mathbf{Y}_{(p)}, \mathbf{Y}_{(q)}\right] = 0. \tag{5.18}$$

But our recursion relations imply stronger constraints. From (5.8)

$$\mathbf{X}_{(r+1,2p)} = \frac{1}{2}\begin{vmatrix} I_{(r)} & K_{(r)} \\ K_{(r)} & I_{(r)} \end{vmatrix} \otimes \mathbf{X}_{(r,2p)} + \frac{1}{2}\begin{vmatrix} I_{(r)} & K_{(r)} \\ -K_{(r)} & -I_{(r)} \end{vmatrix} \otimes \mathbf{Y}_{(r,2p-1)} \tag{5.19}$$

Note that

$$\begin{vmatrix} I_{(r)} & K_{(r)} \\ K_{(r)} & I_{(r)} \end{vmatrix} \begin{vmatrix} I_{(r)} & K_{(r)} \\ -K_{(r)} & -I_{(r)} \end{vmatrix} = 0. \tag{5.20}$$

Along with recursion relations, systematically exploiting the constraints (A.2-12) of Appendix, one obtains that not only do $\mathbf{X}_{(p)}$, $\mathbf{X}_{(q)}$ commute for each r but

$$\mathbf{X}_{(p)}\mathbf{X}_{(q)} = \mathbf{X}_{(q)}\mathbf{X}_{(p)} = 0, \qquad p \neq q. \tag{5.21}$$

Analogously one can show

$$\mathbf{Y}_{(p)}\mathbf{Y}_{(q)} = \mathbf{Y}_{(q)}\mathbf{Y}_{(p)} = 0, \qquad p \neq q. \tag{5.22}$$

These are indeed sufficient and necessary conditions for the eigenvalue spectrum derived for $r = 1, 2, 3, 4$ in sec. 4.

Let $|\mathbf{v}_p\rangle$ denote an eigenstate of $\mathbf{X}_{(p)}$ with the eigenvalue $\mathbf{X}_{(p)}\,|\mathbf{v}_p\rangle = v_p\,|\mathbf{v}_p\rangle$ $(v_p \neq 0)$. Then for $q \neq p$,

$$\mathbf{X}_{(q)}\,|\mathbf{v}_p\rangle = v_p^{-1}\mathbf{X}_{(q)}\mathbf{X}_{(p)}\,|\mathbf{v}_p\rangle = 0. \tag{5.23}$$

Hence

$$(A + D)_r\,|\mathbf{v}_p\rangle = (1 + \mathbf{x})^{r-2p}\,(1 - \mathbf{x})^{2p}\,v_p\,|\mathbf{v}_p\rangle = 0, \qquad p = 0, 2, \ldots. \tag{5.24}$$

But there is still one more class of constraints. Starting with (see (5.2))

$$\mathrm{Tr}\,(\mathbf{T}_1) = \mathrm{Tr}\left((A + D)_1\right) = 2\,(1 + \mathbf{x}) \tag{5.25}$$

and noting that (see (5.8))

$$\mathrm{Tr}\,(\mathbf{T}_{r+1}) = \mathrm{Tr}\left((A + D)_{r+1}\right) = (1 + \mathbf{x})\,\mathrm{Tr}\left((A + D)_r\right), \tag{5.26}$$

one obtains

$$\mathrm{Tr}\left(\mathbf{T}_r\right) = \mathrm{Tr}\left(\left(A+D\right)_r\right) = 2\left(1+\mathbf{x}\right)^r. \tag{5.27}$$

Hence

$$\sum_{p\neq 0} v_p = 0. \tag{5.28}$$

How is this constraint implemented for each r? In our examples (sec. 4, $r = 1,2,3,4$) we saw that

1. v_0 has a multiplicity 2, saturating (5.27).
2. v_p ($p \neq 0$) comes with multiplicity, each case providing a subset of zero sum. For each p one has, one or more, subsets

$$\sum_i v_p^{(i)} = 0. \tag{5.29}$$

3. Here r-th roots of unity play a crucial role. Typically in (5.29) one has

$$\left(1+\mathbf{x}\right)^{r-2p}\left(1-\mathbf{x}\right)^{2p}\left(\sum_{k=0}^{r-1} e^{\mathbf{i}\frac{2\pi}{r}k}\right) = 0. \tag{5.30}$$

4. For $r = 3$ one has only cube roots of unity. For $r = 4$ one has both 2-plets and 4-plets, (square roots of unity being also fourth roots).

Let us now consider possible submultiplets from a more general point of view. The even and odd subspaces introduced in sec. 4 (even and odd multiplicities of the index 1 distinguishing them) can be generalized to all r, each one closed under the action of $\left(A+D\right)_r$ and of dimension 2^{r-1}. In each one there is exactly 1 state with eigenvalue $\left(1+\mathbf{x}\right)^r$ saturating (5.27). Hence one can now consider separately two base spaces of dimension $(2^{r-1}-1)$. When r is a prime number (say L) there is a relative simplicity concerning the multiplet structure. A Theorem of Fermat (see Ref. 1, Appendix B: Encounter with a theorem of Fermat) adapted to our case assures

$$2^{L-1} - 1 = l \cdot L, \tag{5.31}$$

where l is an integer. Thus for $L = 3$, 5, 7, 11, etc., $l = 1,3,9,93$, etc.. Hence an integer number of L-plets can span adequately the $2^{L-1}-1$ dimensional space with $\sum_{k=0}^{r-1} e^{\mathbf{i}\frac{2\pi}{r}k} = 0$. When r is not a prime number each prime factor of r ($4 = 2\times 2$, $6 = 2\times 3$, etc.) can lead to submultiplets with zero sum. Finally, if a singlet occurs in $\{e\}$ the even subspace (i.e. apart from $\left(1+\mathbf{x}\right)^r$) it must occur in $\{o\}$ the odd one with an opposite sign (ex: $\left(1-\mathbf{x}\right)^4$ in $\{e\}$ and $-\left(1-\mathbf{x}\right)^4$ in $\{o\}$ for $r = 4$). The number of possibilities increase rapidly with r. Our study remains incomplete concerning the precise number of multiplets and the multiplicity for each higher r. We have however delineated completely, for all r the dependence of the eigenvalue spectrum on $\mathbf{x}$ (or the spectral parameter θ).

Leu us note one point. We changed over from $\hat{R}(\theta)$ to $\hat{R}(\mathbf{x})$ in sec. 3. Thus gives conveniently a single parameter $\mathbf{x}$ on the anti-diagonal. With the original normalization

$$(1+\mathbf{x})^{r-2p}(1-\mathbf{x})^{2p} \approx e^{(r-2p)m_{11}^{(+)}\theta+2pm_{11}^{(-)}\theta}. \tag{5.32}$$

Finally, starting with $\mathbf{T}_1$ or $\mathbf{T}_2$ (of (4.8)) and implementing recursions one can show that the sum of the elements in each row (and each column) of T_r is $(1+\mathbf{x})^r$. The sum of basic components of $\{e\}$ and $\{o\}$ ($|e_1\rangle$, $|o_1\rangle$ for $r=3,4$ in sec. 4 and their direct generalizations) thus each corresponds to the eigenvalue $(1+\mathbf{x})^r$. The sum of these two furnishes the total trace of $\mathbf{T}_r$, namely $2(1+\mathbf{x})^r$. All the other states together contribute zero trace. Moreover, since for our choice of domains (sec. 3) always $0<\mathbf{x}<1$, $(1+\mathbf{x})>1$, $(1-\mathbf{x})<1$ assuming roots of unity in (5.24) for v_p. $(1+\mathbf{x})^r$ is the largest eigenvalue. This is significant in statistical models.

In this context one should note that the special status of $\mathbf{X}_{(r,0)}$ in the iterative structure. From (5.19) one obtains the matrices structure

$$2\mathbf{X}_{(r+1,0)} = \begin{vmatrix} \mathbf{X}_{(r,0)} & K_{(r)}\mathbf{X}_{(r,0)} \\ K_{(r)}\mathbf{X}_{(r,0)} & \mathbf{X}_{(r,0)} \end{vmatrix} \tag{5.33}$$

Thus, in each subspace, one can construct iteratively exclusively $\mathbf{X}_{(r,0)}$ staring from $\mathbf{X}_{(1,0)}$ and the corresponding 2^{r-1} mutually orthogonal eigenstates:

1. One with eigenvalue $(1+\mathbf{x})^r$.
2. $2^{r-1}-1$ with eigenvalue zero.

The latter provide non-zero eigenvalues for $\mathbf{X}_{(p)}$ with p non zero. One thus obtains the complete basis of eigenstates.

6 Generalizations $(n \geq 2)$

One has for all n

$$\begin{aligned} \mathrm{P}P_{ij}^{(\epsilon)} &= \frac{1}{2}\left\{(ji)\otimes(ij)+(\bar{j}\bar{i})\otimes(\bar{i}\bar{j})+\epsilon\left[(j\bar{i})\otimes(i\bar{j})+(\bar{j}i)\otimes(\bar{i}j)\right]\right\}, \\ \mathrm{P}P_{i\bar{j}}^{(\epsilon)} &= \frac{1}{2}\left\{(\bar{j}i)\otimes(i\bar{j})+(j\bar{i})\otimes(\bar{i}j)+\epsilon\left[(\bar{j}\bar{i})\otimes(ij)+(ji)\otimes(\bar{i}\bar{j})\right]\right\}. \end{aligned} \tag{6.1}$$

These lead (for fundamental blocks or $r=1$) to

$$\begin{aligned} \mathbf{T}_{ij} &= a_{ji}^{(+)}(ji)+a_{ji}^{(-)}(\bar{j}\bar{i}), & \mathbf{T}_{\bar{i}\bar{j}} &= a_{ji}^{(+)}(\bar{j}\bar{i})+a_{ji}^{(-)}(ji), \\ \mathbf{T}_{i\bar{j}} &= a_{ji}^{(-)}(j\bar{i})+a_{ji}^{(+)}(\bar{j}i), & \mathbf{T}_{\bar{i}j} &= a_{ji}^{(-)}(\bar{j}i)+a_{ji}^{(+)}(j\bar{i}), \end{aligned} \tag{6.2}$$

where $a_{ij}^{(\pm)}=\frac{1}{2}\left(e^{m_{ij}^{(+)}\theta}\pm e^{m_{ij}^{(-)}\theta}\right)$. For $m_{ij}^{(+)}>m_{ij}^{(-)}$ (resp. $m_{ij}^{(+)}<m_{ij}^{(-)}$) all Boltzmann weights are nonnegative for $\theta>0$ (resp. $\theta<0$). Note that the fundamental blocks are now

$(2n \times 2n)$ matrices with only two non-zero elements each. This number does not change with n. In a compact notation with indices $a \in \{1, 2, \dots, 2n\}$ and $\bar{a} = \{2n, 2n-1, \dots, 1\}$ correspondingly, one can write, for $r = 1$,

$$\mathbf{T}_{ab} = a_{ba}^{(+)}\,(ba) + a_{ba}^{(-)}\,\left(\bar{b}\bar{a}\right) \tag{6.3}$$

with (not only $a_{ij}^{(\pm)} = a_{\bar{i}\bar{j}}^{(\pm)}$, but)

$$a_{ba}^{(\pm)} = a_{\bar{b}\bar{a}}^{(\pm)} = a_{b\bar{a}}^{(\pm)} = a_{\bar{b}a}^{(\pm)}. \tag{6.4}$$

The coproduct rules gives the iterative structure

$$\mathbf{T}_{ab}^{(r+1)} = \sum_c \mathbf{T}_{ac} \otimes \mathbf{T}_{cb}^{(r)} = \sum_c \left(a_{ca}^{(+)}\,(ca) + a_{ca}^{(-)}\,(\bar{c}\bar{a})\right) \otimes \mathbf{T}_{cb}^{(r)}. \tag{6.5}$$

The fact that, as in (6.3), only diagonal blocks have diagonal elements can be easily shown to lead to

$$\Im_r \equiv \mathrm{Tr}\left(\mathbf{T}^{(r)}\right) = 2\left(\sum_{i=1}^{n} e^{r m_{ii}^{(+)}\theta}\right). \tag{6.6}$$

This is the direct multiparametric generalization of the 4×4 case

$$\Im_r = 2e^{r m_{11}^{(+)}\theta}. \tag{6.7}$$

It is instructive to study the case $n = 2$ explicitly. Denoting (as in Ref. 1 with $\bar{1} = 2$, $\bar{2} = 1$)

$$\begin{aligned} a_{(\pm)} &= \frac{1}{2}\left(e^{m_{11}^{(+)}\theta} \pm e^{m_{11}^{(-)}\theta}\right), \qquad & d_{(\pm)} &= \frac{1}{2}\left(e^{m_{22}^{(+)}\theta} \pm e^{m_{22}^{(-)}\theta}\right), \\ b_{(\pm)} &= \frac{1}{2}\left(e^{m_{12}^{(+)}\theta} \pm e^{m_{12}^{(-)}\theta}\right), \qquad & c_{(\pm)} &= \frac{1}{2}\left(e^{m_{21}^{(+)}\theta} \pm e^{m_{21}^{(-)}\theta}\right) \end{aligned} \tag{6.8}$$

for $r = 1$ (with now $i = 1, 2$, $\bar{i} = 4, 3$ below - a change of notation convenient for displaying symmetries) one has

$$\mathbf{T}_{11} = \begin{vmatrix} a_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_- \end{vmatrix}, \qquad \mathbf{T}_{22} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & d_+ & 0 & 0 \\ 0 & 0 & d_- & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$\mathbf{T}_{1\bar{1}} = \begin{vmatrix} 0 & 0 & 0 & a_- \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_+ & 0 & 0 & 0 \end{vmatrix}, \qquad \mathbf{T}_{2\bar{2}} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & d_- & 0 \\ 0 & d_+ & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$\mathbf{T}_{12} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ c_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & c_- \\ 0 & 0 & 0 & 0 \end{vmatrix}, \qquad \mathbf{T}_{1\bar{2}} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_- \\ c_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$T_{21} = \begin{vmatrix} 0 & b_+ & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b_- & 0 \end{vmatrix}, \qquad \mathbf{T}_{2\bar{1}} = \begin{vmatrix} 0 & 0 & b_- & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b_+ & 0 & 0 \end{vmatrix}. \tag{6.9}$$

The remaining 8 blocks are given by

$$(a_+, b_+, c_+, d_+) \rightleftharpoons (a_-, b_-, c_-, d_-) \qquad \Longrightarrow \qquad T_{ab} \rightleftharpoons T_{\bar{a}\bar{b}}. \tag{6.10}$$

One has

$$\Im_1 = \mathrm{Tr}\left(\mathbf{T}_{11} + \mathbf{T}_{22} + \mathbf{T}_{\bar{1}\bar{1}} + \mathbf{T}_{\bar{2}\bar{2}}\right) = 2\left((a_+ + a_-) + (d_+ + d_-)\right) = 2\left(e^{m_{11}^{(+)}\theta} + e^{m_{22}^{(+)}\theta}\right). \tag{6.11}$$

The recursion

$$\Im_{r+1} = \mathrm{Tr}\left((a_+ + a_-)\left(\mathbf{T}_{11}^{(r)} + \mathbf{T}_{\bar{1}\bar{1}}^{(r)}\right) + (d_+ + d_-)\left(\mathbf{T}_{22}^{(r)} + T_{\bar{2}\bar{2}}^{(r)}\right)\right) \tag{6.12}$$

leads to

$$\Im_r = 2\left(e^{rm_{11}^{(+)}\theta} + e^{rm_{22}^{(+)}\theta}\right) \tag{6.13}$$

a particular case of (6.6). For $n = 2$ the diagonal blocks, for example, have the iterative structures

$$\mathbf{T}_{11}^{(r+1)} = \begin{vmatrix} a_+\mathbf{T}_{11}^{(r)} & 0 & 0 & a_-\mathbf{T}_{\bar{1}\bar{1}}^{(r)} \\ c_+\mathbf{T}_{21}^{(r)} & 0 & 0 & c_-\mathbf{T}_{\bar{2}\bar{1}}^{(r)} \\ c_+\mathbf{T}_{\bar{2}\bar{1}}^{(r)} & 0 & 0 & c_-\mathbf{T}_{21}^{(r)} \\ a_+\mathbf{T}_{\bar{1}\bar{1}}^{(r)} & 0 & 0 & a_-\mathbf{T}_{11}^{(r)} \end{vmatrix} \qquad \mathbf{T}_{22}^{(r+1)} = \begin{vmatrix} 0 & b_+\mathbf{T}_{12}^{(r)} & b_-\mathbf{T}_{\bar{1}\bar{2}}^{(r)} & 0 \\ 0 & d_+\mathbf{T}_{22}^{(r)} & d_-\mathbf{T}_{\bar{2}\bar{2}}^{(r)} & 0 \\ 0 & d_+\mathbf{T}_{\bar{2}\bar{2}}^{(r)} & d_-\mathbf{T}_{22}^{(r)} & 0 \\ 0 & b_+\mathbf{T}_{\bar{1}\bar{2}}^{(r)} & b_-\mathbf{T}_{12}^{(r)} & 0 \end{vmatrix} \tag{6.14}$$

$\mathbf{T}_{\bar{1}\bar{1}}^{(r+1)}$ and $\mathbf{T}_{\bar{2}\bar{2}}^{(r+1)}$ are now obtained by setting respectively in $\mathbf{T}_{11}^{(r+1)}$ and $\mathbf{T}_{22}^{(r+1)}$ $a_-\mathbf{T}_{\bar{1}\bar{1}}^{(r)}$ for $a_+\mathbf{T}_{11}^{(r)}$ and so on, systematically in an evident fashion. Their sum gives the transfer matrix of order $(r+1)$ exhibiting the iterative structure. For $r = 1$ the transfer matrix is directly diagonal for all n giving directly the eigenvalues. For $n = 2$, for example (consistently with (6.13))

$$\mathbf{T}^{(1)} = \begin{vmatrix} e^{m_{11}^{(+)}\theta} & 0 & 0 & 0 \\ 0 & e^{m_{22}^{(+)}\theta} & 0 & 0 \\ 0 & 0 & e^{m_{22}^{(+)}\theta} & 0 \\ 0 & 0 & 0 & e^{m_{11}^{(+)}\theta} \end{vmatrix}, \tag{6.15}$$

with evident generalization for $n > 2$. For $n = 1$ we have systematically explored the remarkable structure of the transfer matrix for all r (see (5.14-18)) and consequences for eigenstates. A parallel study for $n > 1$ is beyond the scope of this paper. Our results in the section already indicate how the multiparametric aspects start playing on essential role.

In sec. 5, starting with the 2×2 matrix K (5.3-7) at the level of $r=1$ and implementing tensor products, powerful recursion relations were obtained. We started by relating (A,B,C,D) among themselves. For $N\geq 2$ one can similarly relate (for a given pair of indices (i,j)) the quartet $\left(\mathbf{T}_{ij},\mathbf{T}_{\bar{i}\bar{j}},\mathbf{T}_{i\bar{j}},\mathbf{T}_{\bar{i}j}\right)$ given, for $r=1$, by (6.2). Evidently one can relate through constant matrices only blocks involving the same pair of parameters $\left(m_{ij}^{(\pm)}\right)$. For this one introduces the matrix

$$\sum_{i=1}^{n}\left((i\bar{i})+(\bar{i}i)\right) \tag{6.16}$$

generalizing $K=(1\bar{1})+(\bar{1}1)=\begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix}$. Thus for $n=2$, one has

$$K\otimes K=\begin{vmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{vmatrix}=K^{(2)}. \tag{6.17}$$

The generalization is evident. From (6.8)

$$\begin{aligned}
&K^{(2)}\left(\mathbf{T}_{11},\mathbf{T}_{\bar{1}\bar{1}},\mathbf{T}_{1\bar{1}},\mathbf{T}_{\bar{1}1}\right)=\left(\mathbf{T}_{1\bar{1}},\mathbf{T}_{\bar{1}1},\mathbf{T}_{11},\mathbf{T}_{\bar{1}\bar{1}}\right),\\
&K^{(2)}\left(\mathbf{T}_{11},\mathbf{T}_{\bar{1}\bar{1}},\mathbf{T}_{1\bar{1}},\mathbf{T}_{\bar{1}1}\right)K^{(2)}=\left(\mathbf{T}_{\bar{1}\bar{1}},\mathbf{T}_{11},\mathbf{T}_{\bar{1}1},\mathbf{T}_{1\bar{1}}\right)
\end{aligned} \tag{6.18}$$

with exactly analogous results for other subsets. For each n one has, in evident notations, for fixed (i,j)

$$\begin{aligned}
&K^{(n)}\left(\mathbf{T}_{ij},\mathbf{T}_{\bar{i}\bar{j}},\mathbf{T}_{i\bar{j}},\mathbf{T}_{\bar{i}j}\right)=\left(\mathbf{T}_{i\bar{j}},\mathbf{T}_{\bar{i}j},\mathbf{T}_{ij},\mathbf{T}_{\bar{i}\bar{j}}\right),\\
&K^{(n)}\left(\mathbf{T}_{ij},\mathbf{T}_{\bar{i}\bar{j}},\mathbf{T}_{i\bar{j}},\mathbf{T}_{\bar{i}j}\right)K^{(n)}=\left(\mathbf{T}_{\bar{i}\bar{j}},\mathbf{T}_{ij},\mathbf{T}_{\bar{i}j},\mathbf{T}_{i\bar{j}}\right).
\end{aligned} \tag{6.19}$$

We will not attempt to explore in the present paper the applications of such relations generalizing our results of sec. 5.

7 Spin chain Hamiltonians

Our construction of odd dimensional Hamiltonians (sec. 4, Ref. 1) can be adapted to the present even dimensional cases as follows. One has, taking derivatives and setting $\theta=0$, for $n=1$,

$$\dot{\hat{R}}(0)=\begin{vmatrix}\mathbf{x}_+ & 0 & 0 & \mathbf{x}_-\\ 0 & \mathbf{x}_+ & \mathbf{x}_- & 0\\ 0 & \mathbf{x}_- & \mathbf{x}_+ & 0\\ \mathbf{x}_- & 0 & 0 & \mathbf{x}_+\end{vmatrix}, \tag{7.1}$$

where $\mathbf{x}_\pm = \frac{1}{2}\left(m_{11}^{(+)} \pm m_{11}^{(-)}\right)$ and $\dot{\hat{R}}(0)$ is obtained by setting $\theta = 0$ in $\frac{d}{d\theta}\hat{R}(\theta)$. For $n = 2$, setting (starting from (6.8))

$$\begin{aligned} \hat{a}_\pm &= \frac{1}{2}\left(m_{11}^{(+)} \pm m_{11}^{(-)}\right), \qquad & \hat{d}_\pm &= \frac{1}{2}\left(m_{22}^{(+)} \pm m_{22}^{(-)}\right), \\ \hat{b}_\pm &= \frac{1}{2}\left(m_{12}^{(+)} \pm m_{12}^{(-)}\right), \qquad & \hat{c}_\pm &= \frac{1}{2}\left(m_{21}^{(+)} \pm m_{21}^{(-)}\right) \end{aligned} \tag{7.2}$$

from (2.7) and

$$\dot{\hat{R}}(0) = \begin{vmatrix} \hat{D}_{11} & 0 & 0 & \hat{A}_{1\bar{1}} \\ 0 & \hat{D}_{22} & \hat{A}_{2\bar{2}} & 0 \\ 0 & \hat{A}_{\bar{2}2} & \hat{D}_{\bar{2}\bar{2}} & 0 \\ \hat{A}_{\bar{1}1} & 0 & 0 & \hat{D}_{\bar{1}\bar{1}} \end{vmatrix}, \tag{7.3}$$

where

$$\begin{aligned} \hat{D}_{11} &= \hat{D}_{\bar{1}\bar{1}} = \left(\hat{a}_+, \hat{b}_+, \hat{b}_+, \hat{a}_+\right)_{\text{diag.}} \qquad & \hat{D}_{22} &= \hat{D}_{\bar{2}\bar{2}} = \left(\hat{c}_+, \hat{d}_+, \hat{d}_+, \hat{c}_+\right)_{\text{diag.}} \\ \hat{A}_{1\bar{1}} &= \hat{A}_{\bar{1}1} = \left(\hat{a}_-, \hat{b}_-, \hat{b}_-, \hat{a}_-\right)_{\text{anti-diag.}} \qquad & \hat{A}_{2\bar{2}} &= \hat{A}_{\bar{2}2} = \left(\hat{c}_-, \hat{d}_-, \hat{d}_-, \hat{c}_-\right)_{\text{anti-diag.}} \end{aligned} \tag{7.4}$$

The extension of our formalism for $n > 2$ is straightforward.

For r sites the standard result for the Hamiltonian is (see sources cited in Ref. 1)

$$\mathrm{H} = \sum_{k=1}^{r} I \otimes \cdots \otimes \dot{\hat{R}}(0)_{k,k+1} \otimes \cdots \otimes I, \tag{7.5}$$

where for circular boundary conditions $k + 1 = r + 1 \approx 1$. We intend to present a more complete study of our spin chain elsewhere. But here one already sees how the two aspects, multistate (higher spins at each site) and multiparameter $\left(m_{ij}^{(\pm)}\right)$ get directly associated for our hierarchy. In our previous papers [3, 6] and here again (see (3.4), (3.5)) we showed how the passage to imaginary parameters can lead to unitary $\hat{R}(\theta)$. The corresponding $\dot{\hat{R}}(0)$ has only an overall factor $\mathbf{i}$ which can be extracted from the sum (7.5).

8 Remarks

I. Status of eigenstates for $n = 1$**:** For the simplest 4×4 braid matrix in our hierarchy the construction of the eigenvalues and eigenfuntions of transfer matrices of successive orders ($r = 1$, 2, 3, 4, etc.) has attained the following stage:

(1) The transfer matrix at the level r has been expressed in the form of a $2^r \times 2^r$ matrix

$$\mathbf{T}_r = \sum_{p=0,1,2,\ldots,p_m} (1 + \mathbf{x})^{r-2p} (1 - \mathbf{x})^{2p} \mathbf{X}_{(p)}, \tag{8.1}$$

where $\mathbf{x} = \tanh\frac{1}{2}\left(m_{11}^{(+)} - m_{11}^{(-)}\right)\theta$ and $2p_m = r$ (resp. $r-1$) for r even (resp. odd) $(2p_m = r - (1-(-1)^r)/2)$. θ being the spectral parameter, $m_{11}^{(\pm)}$ two free parameters (see (3.3)). The matrices $\mathbf{X}_{(p)}$ are constant ones ($\mathbf{x}$-independent) and satisfy

$$\mathbf{X}_{(p)}\mathbf{X}_{(q)} = \mathbf{X}_{(q)}\mathbf{X}_{(p)} = 0, \qquad p \neq q. \tag{8.2}$$

They can be computed systematically via the recursion relations (for $r \longrightarrow r+1$).

(2) For

$$\begin{aligned} &\mathbf{X}_{(p)}\left|p_{(i)}\right\rangle = v_{(p,i)}\left|p_{(i)}\right\rangle, \qquad \left(v_{(p,i)} \neq 0\right), \\ &\mathbf{X}_{(q)}\left|p_{(i)}\right\rangle = 0, \qquad p \neq q \end{aligned} \tag{8.3}$$

and $v_{(p,i)}$ denote phase factors which come in multiplets of zero sum formed by roots of unity corresponding to r and its prime factors (see examples in sec. 4). The index "i" denotes such possible multiplicity of each p. The exception of the zero sum rule corresponds to $p=0$. One obtains for each r, twice $v_{(0)} = 1$ giving

$$\mathrm{Tr}\left(\mathbf{T}_r\right) = 2\left(1+\mathbf{x}\right)^r \tag{8.4}$$

a general constraint obtained via recursions. This multiplicity 2 corresponds to two 2^{r-1} dimensional subspaces (see "even", "odd" subspaces defined in sec. 4) each providing just one eigenstate of $\mathbf{X}_{(0)}$ with non-zero eigenvalue $(1+\mathbf{x})^r$.

(3) Thus the problem has been reduced to construction of eigenstates of each $\mathbf{X}_{(p)}$ separately, reducing the dimension by considering each (even, odd) subspace by turn. This involves solving sets of linear constraints with only positive and negative integers as coefficients. One finally keeps only the non-zero eigenvalues for each p , they being associated with zero eigenvalues for $\mathbf{X}_{(q)}$, $q \neq p$. In fact as noted below (5.33), it suffices to construct the full sect of mutually orthogonal eigenstates of $\mathbf{X}_{(r,0)}$, all but one in each subspace having eigenvalue zero.

(4) As already stated (sec. 5) our results remain incomplete concerning the pattern of possible multiplets and submultiplets corresponding to roots of unity provided by r and its prime factors - and multiplicities of such multiplets. A canonical enumeration when r has a very large number of prime factors seems to be an unlikely possibility. Nor have we established rigorously that $v_{(p,i)}$ in (8.3) are always ± 1 or higher roots of unity phases factors. This what happens in examples ($r \leq 4$) of sec. 4 and directly leads to the following obligatory constraint (8.4).

(4) We have completely, and for all r, extracted the θ-dependence of eigenvalues in (8.1).

II. Comparisons with standard six vertex and eight vertex models: From (2.5) and (3.1) our Yang-Baxter matrix (for $n = 1$) with corresponding normalizations, is

$$R(\theta) = \mathrm{P}\hat{R}(\theta) = \begin{vmatrix} a_+ & 0 & 0 & a_- \\ 0 & a_- & a_+ & 0 \\ 0 & a_+ & a_- & 0 \\ a_- & 0 & 0 & a_+ \end{vmatrix}, \tag{8.5}$$

where $a_\pm = \frac{1}{2}\left(e^{m_{11}^{(+)}\theta} \pm e^{m_{11}^{(-)}\theta}\right)$ and, equivalently,

$$R(\mathbf{x}) = \begin{vmatrix} 1 & 0 & 0 & \mathbf{x} \\ 0 & \mathbf{x} & 1 & 0 \\ 0 & 1 & \mathbf{x} & 0 \\ \mathbf{x} & 0 & 0 & 1 \end{vmatrix}, \tag{8.6}$$

where $\mathbf{x} = \tanh\frac{1}{2}\left(m_{11}^{(+)} - m_{11}^{(-)}\right)\theta$. Let us now compare this to the very well known 4×4 six vertex and eight vertex models - concerning which it is sufficient to a cite a standard text book [7] and review articles [5, 8] which cite basic sources. All such cases are of the form

$$R(\theta) = \begin{vmatrix} a & 0 & 0 & d \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ d & 0 & 0 & a \end{vmatrix}, \tag{8.7}$$

For (8.6) $a = c = 1$, $b = d = \mathbf{x}$ and for (8.5) $a = c = a_+$, $b = d = a_-$. For six vertex, crucially, $d = 0$ and (a, b, c) is being given, according to the regime, by circular or hyperbolic functions. For eight vertex, famously, elliptic functions appear and $d \neq 0$. For our case $d \neq 0$ but arguably, one has maximal simplicity and symmetry compatible with non-trivial solution for a 4×4 Yang-Baxter (or braid) matrix. We have all eight vertices but (with, say (8.5)) (see fig. 1) corresponding to $a = c = a_+$ and (see fig. 2)

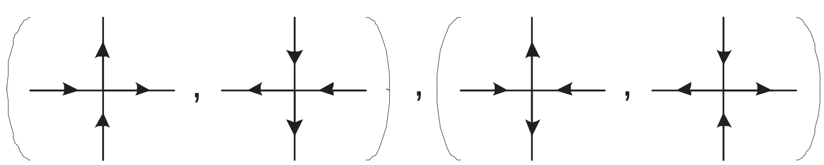

fig. 1

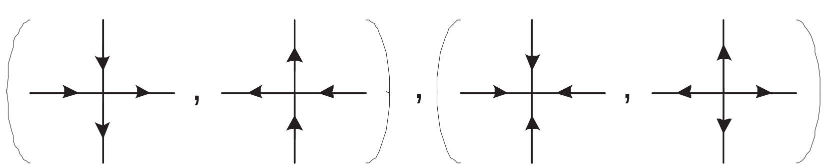

fig. 2

corresponding to $b = d = a_-$. In six vertex the last two vertices are excluded $(d = 0)$. We will not study, in this paper, the implications of the results below (8.7) concerning various properties of our model (compare the relevant detailed study of eight vertex in Ref. 7). But we would like to contrast our approach to the construction of eigenstates and extraction of eigenvalues of $\mathbf{T}_r$ with that via Bethe ansatz in standard six vertex models [5]. The systematic study of $R\mathbf{TT}$ constraints are particularly relevant (see Appendix A).

In six vertex the Bethe ansatz construction involves pushing $(A(\theta) + D(\theta))$ through the product $B(\theta_1)\, B(\theta_2) \ldots B(\theta_r)$ acting on one single state $\left|\begin{matrix}1\\0\end{matrix}\right\rangle_1 \otimes \left|\begin{matrix}1\\0\end{matrix}\right\rangle_2 \otimes \cdots \left|\begin{matrix}1\\0\end{matrix}\right\rangle_r \equiv |11\ldots 1\rangle_r$ and eliminating unwanted terms to obtain a complete set of eigenstates corresponding to the sets of resulting constraints. This involves solving nonlinear equations. Our resorts to programs and numerical studies for higher r's.

In our case recursion relations for $(A \pm D)_r$ (for $r \longrightarrow r+1$) are sufficient to attain the stage systematically presented in part (I) of this section. One solves , at each stage, linear equations with integer coefficients. In fact since each r (see (5.5-7)) $(B \pm C)_r = K_{(r)}\,(A \pm D)_r$, where $K_{(r)}$ in the $2^r \times 2^r$ matrix with 2^{r-1}-times $K = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ on diagonal , the actions of $(B \pm C)_r$ follows immediately from those of $(A \pm D)_r$. They need hardly be studied separately. However that $K_{(r)}$ and hence $(B \pm C)_r$ connect the even and odd subspaces. On the other hand $(A + D)$ cannot be pushed through a product of $B(\theta)$'s displayed above. The nearest approaches are typically our (A.10) and (A.12) (changes of signs in $(A \pm D)$ and $(B \pm C)$ are to be noted as $(A + D)$ is pushed through). Subtle analytic properties, unlike six vertex and particularly eight vertex models, play no role in our case. One has only to look at the x (or θ) dependence in (8.1).

In a simple situation the contrast between our model and the standard eight vertex one shows up very clearly: For the asymptotic case θ tending to infinity one sets $\mathbf{x} = 1$ in (8.6) to obtain

$$R = \hat{R} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix}. \tag{8.8}$$

Whereas, choosing adequately the normalizing factor (see sec. 7 of [9] and in particular eq. (7.13)) for the standard case the corresponding limiting form is

$$R = \left(1, q^{-1}, q^{-1}, 1\right)_{\text{diag.}} \neq \hat{R}. \tag{8.9}$$

When diagonalized (or block diagonalized) our classes of matrices, in general, lose braid (or Yang-Baxter) property unless the diagonalizer has a corrected tensor structure (sec. 4 and Addendum of ref. 3).

III. passage to higher dimensions $n \geq 2$**:** As already emphasized before, a major interest of our simple model for $n = 1$ (the 4×4 braid matrix) is that it is the first one in a hierarchy of $(2n)^2 \times (2n)^2$ braid matrices with $2n^2$ free parameters at each level. Some

of the simplicity of the $n=1$ case is inevitably lost as n increases. But we have pointed out in sections 6 and 7 how certain basic features vary in a simple, canonical fashion. Thus for example,

(1)

$$Tr\left(\mathbf{T}^{(r)}\right)=2\left(\sum_{i=1}^{n} e^{rm_{ii}^{(+)}\theta}\right), \qquad \text{for all } n. \tag{8.10}$$

(2) Each blocks $\mathbf{T}_{ij}$ of the $\mathbf{T}$ matrix has just two non-zero elements (see (6.2) and (6.9)) out of $(2n \times 2n)$, namely $a_{ji}^{(\pm)}(\theta)$ for all n.

(3) For a fixed pair of indices (i,j) the blocks $\left(\mathbf{T}_{ij}, \mathbf{T}_{\bar{i}\bar{j}}, \mathbf{T}_{i\bar{j}}, \mathbf{T}_{\bar{i}j}\right)$ can be quite simply related among themselves (see (6.18)) via direct generalizations of the matrix $K = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ for the 4×4 case. For $n=1$ such relations led to recursion relations yielding (8.1-3).

(4) Spin chain Hamiltonians present (see eqs. (7.1-5)) a simple canonical sequence as n increases.

We hope to study the higher dimensional cases more fully elsewhere.

Acknowledgments: *One of us (BA) wants to thank Pierre Collet and Paul Sorba for precious help.*

Appendix A
RTT constraints $(n = 1)$

From (3.10) one obtains with K from (3.9)

$$\begin{aligned}
&\begin{vmatrix} A & B \\ C & D \end{vmatrix} \otimes \begin{vmatrix} A' & B' \\ C' & D' \end{vmatrix} - \begin{vmatrix} A' & B' \\ C' & D' \end{vmatrix} \otimes \begin{vmatrix} A & B \\ C & D \end{vmatrix} \\
&= \mathbf{x}'' \left(\begin{vmatrix} B' & A' \\ D' & C' \end{vmatrix} \otimes \begin{vmatrix} B & A \\ D & C \end{vmatrix} - \begin{vmatrix} C & D \\ A & B \end{vmatrix} \otimes \begin{vmatrix} C' & D' \\ A' & B' \end{vmatrix} \right) \\
&= \mathbf{x}'' \left(\begin{vmatrix} B & A \\ D & C \end{vmatrix} \otimes \begin{vmatrix} B' & A' \\ D' & C' \end{vmatrix} - \begin{vmatrix} C' & D' \\ A' & B' \end{vmatrix} \otimes \begin{vmatrix} C & D \\ A & B \end{vmatrix} \right) .
\end{aligned} \tag{A.1}$$

The last step follows from $\mathbf{x} \rightleftharpoons \mathbf{x}'$ on both sides of the first two expressions since under this interchange (see (3.7)) $\mathbf{x}'' \longrightarrow -\mathbf{x}''$. The consistency of the last two implies that each element of the total matrix on the right (apart from the factor $\mathbf{x}''$) must be symmetric in $(\mathbf{x}, \mathbf{x}')$. This can indeed be verified starting from $r = 1, 2, 3, \ldots$ using the standard construction in sec. 3.

From the last two steps (the factor $\mathbf{x}''$ cancelling) one obtains, with only upper or lower signs,

$$\begin{aligned}
(A \pm D)\,(A' \pm D') &= (A' \pm D')\,(A \pm D)\,, \\
(B \pm C)\,(B' \pm C') &= (B' \pm C')\,(B \pm C)\,, \\
(A \pm D)\,(B' \pm C') &= (A' \pm D')\,(B \pm C)\,, \\
(B \pm C)\,(A' \pm D') &= (B' \pm C')\,(A \pm D)\,.
\end{aligned} \tag{A.2}$$

One gets 8 relations of the type

$$(M_1 M_2' - M_1' M_2) = \mathbf{x}''\,(M_3' M_4 - M_5 M_6') = \mathbf{x}''\,(M_3 M_4' - M_5' M_6)\,, \tag{A.3}$$

with

$$\begin{aligned}
&(M_1, M_2, M_3, M_4, M_5, M_6) \\
&= \{(A,B,B,A,C,D)\,,(A,C,B,D,C,A)\,,(B,A,A,B,D,C)\,(B,D,A,C,D,B)\,, \\
&(C,A,D,B,A,C)\,,(C,D,D,C,A,B)\,,(D,B,C,A,B,D)\,,(D,C,C,D,B,A)\}\,.
\end{aligned} \tag{A.4}$$

In sec. 5 (see (5.4-7)) we obtained, for any $\mathbf{x}$ and all r,

$$(B, C) = K_{(r)}\,(A, D)\,, \qquad (A, D) = K_{(r)}\,(B, C)\,, \tag{A.5}$$

where ($\mathbf{x}$-independent) matrix $K_{(r)}$ is given. Hence multiplying in (A.3) by $K_{(r)}$ on the left one gets another set with (M_1, M_3, M_5) replaced by $K_{(r)}\,(M_1, M_3, M_5)$, where thus

$$(A, B, B, A, C, D) \longrightarrow (B, B, A, A, D, D) \tag{A.6}$$

and so on. For another class of relations we introduce

$$\mathbf{f}^{(\pm)} = \frac{1}{2}\left(\mathbf{x}'' \pm \frac{1}{\mathbf{x}''}\right) \tag{A.7}$$

From (3.7)

$$\mathbf{f}^{(+)} = \coth\mu\,(\theta-\theta')\,, \qquad \mathbf{f}^{(-)} = -\mathrm{cosech}\mu\,(\theta-\theta') \tag{A.8}$$

One can show that (with upper or lower signs)

$$\begin{aligned}(A \pm D)\,(A' \mp D') &= \mathbf{f}^{(+)}\,(B \pm C)\,(B' \mp C') + \mathbf{f}^{(-)}\,(B' \pm C')\,(B \mp C)\,, & \text{(A.9)}\\ (A \pm D)\,(B' \mp C') &= \mathbf{f}^{(+)}\,(B \pm C)\,(A' \mp D') + \mathbf{f}^{(-)}\,(B' \pm C')\,(A \mp D)\,. & \text{(A.10)}\end{aligned}$$

Again, as for (A.5), (A.6), multiplying the above from the left by $K_{(r)}$ one gets another set of relations such that

$$(B \pm C)\,(B' \pm C') = \mathbf{f}^{(+)}\,(A \pm D)\,(A' \mp D') + \mathbf{f}^{(-)}\,(A' \pm D')\,(A \mp D)\,. \tag{A.11}$$

and so on. For "two-steep" relations redefine x'' with indices $\mathbf{x}_{ij} = \frac{\mathbf{x}_i - \mathbf{x}_j}{1-\mathbf{x}_i\mathbf{x}_j}$ and $\mathbf{f}^{(+)}_{ij} = \coth\mu\,(\theta_i-\theta_j)$, $\mathbf{f}^{(-)}_{ij}$ =-cosech$\mu\,(\theta_i-\theta_j)$ correspondingly. One obtains, for example, for arguments $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$

$$\begin{aligned}(A+D)_1\,(B-C)_2\,(B+C)_3 = {} & \mathbf{f}^{(+)}_{12}\mathbf{f}^{(+)}_{23}\,(B+C)_1\,(B-C)_2\,(A+D)_3 + \\ & \mathbf{f}^{(+)}_{12}\mathbf{f}^{(-)}_{23}\,(B+C)_1\,(B-C)_3\,(A+D)_2 + \\ & \mathbf{f}^{(-)}_{12}\mathbf{f}^{(+)}_{13}\,(B+C)_2\,(B-C)_1\,(A+D)_3 + \\ & \mathbf{f}^{(-)}_{12}\mathbf{f}^{(-)}_{13}\,(B+C)_2\,(B-C)_3\,(A+D)_1\,. \end{aligned} \tag{A.12}$$

See the relevant remarks in sec. 8.

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